Last Time: Inclusion-Exclusion

Pigeonhole Principle

Ex: Let $S = \{n \in \mathbb{Z} : 0 < n \leq 1000 \text{ and } n \text{ is even}\}$.

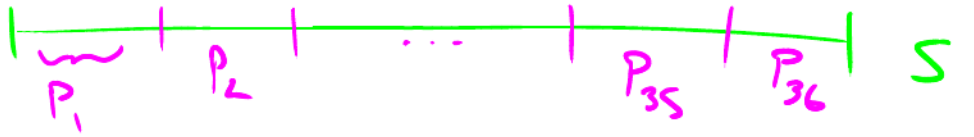
Every 38 elements of S contains a pair $a, b \in S$ distinct with $|a-b| \leq 26$.

Sol: Let $P_k := \{x \in S : 2 + (k-1)28 \leq x \leq 28k\}$

for $1 \leq k \leq 35$ and

$P_{36} = \{x \in S : 982 \leq x\}$

Picture



$\mathcal{P} = \{P_k : 1 \leq k \leq 36\}$ is a weak partition of S with 36 parts. On the other hand, for all $x, y \in P_k$ (for k fixed), we have

$2 + (k-1)28 \leq x, y \leq 28k$

Hence $|x-y| \leq 28k - (28(k-1)+2) = 26$.

Let $T \subseteq S$ have size 38. Because we have 36 parts in $\mathcal{P}$, some two elements of T belong to the same part of $\mathcal{P}$, by the Pigeonhole Principle.

Hence, those two elements of T satisfy $|a-b| \leq 26$ as desired. ☐

Relations and Functions

IDEA: Model relationships between elements of various sets...

Def^n: Let A, B be sets. A relation from A to B is any subset $R \subseteq A \times B$. We often write

* $\left[A \xrightarrow{R} B \right]$ to mean "R is a relation from A to B".

we also write "aRb" to denote $(a,b) \in R$.

Ex: $R = \{(1,4), (2,4), (1,3), (2,5)\} \subseteq [4] \times [7]$,

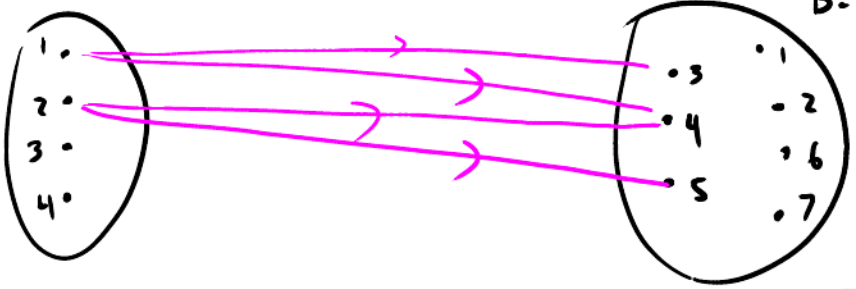
So $[4] \xrightarrow{R} [7]$. $1 R 4$, $1 R 3$.

NB: To represent relations via pictures:

write $A \rightrightarrows B$

{ arrows connect related elements...

$A = [4]$ $\quad$ $B = [7]$



NB: A relation on set S is a relation $S \xrightarrow{R} S$.

Ex: Let $P = \{x : x \text{ is a person}\}$. The following are all relations on P:

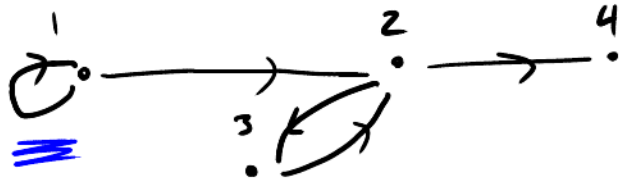
① $P \xrightarrow{sis} P$ where $(x,y) \in sis$ when x is a sister of y.

② $P \xrightarrow{fri} P$ where x fri y when x and y are mutual friends.

③ $P \xrightarrow{\text{Par}} P$ where x Par y when x is a parent of y.

NB: When we study relations on a set, we can simplify our diagrams, like so: just lay out the elements of S and draw arrows between them.

Ex: $R = \{(1,1), (1,2), (2,3), (3,2), (2,4)\} \subseteq [4] \times [4]$.
is a relation on $[4]$. The picture is:



Defn: A relation R on S is reflexive when for all $x \in S$ we have $x R x$.

Ex: The relation from the previous example is not reflexive because $(2,2) \notin R$.

Ex: Consider the relation $x R y$ when $5 \mid (x-y)$ on $\mathbb{N}$.
This relation is reflexive. Indeed, for all $x \in \mathbb{N}$,
we have $x - x = 0 = 5 \cdot 0$, so $5 \mid x-x$ yields xRx.
Hence R is reflexive on $\mathbb{N}$.

Ex: Consider the relation $x R y$ when $5 \mid (x - y + 1)$.
This relation is not reflexive. Let $x = 0$.
$x - x + 1 = 0 - 0 + 1 = 1$, so $5 \nmid (0 - 0 + 1)$; thus
$0 \not R 0$. Hence R is NOT reflexive.
↑ "does not relate via R"

NB: In the pictures, "reflexive" means there is a loop at every element: .

Obs: $x = y$, then $y = x$ ...

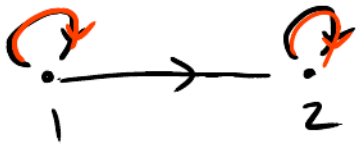
Defn: A relation R on S is symmetric when for all $x, y \in S$ we have xRy implies yRx.

Ex: Let xRy when $5 \mid (x-y)$ on $\mathbb{Z}$. Then R is symmetric. Let $x, y \in \mathbb{Z}$ and suppose xRy. Thus $5 \mid (x-y)$, so $x - y = 5k$ for some $k \in \mathbb{Z}$. Multiplying by -1, we have $y - x = -5k = 5(-k)$ ↑ $-k \in \mathbb{Z}$ by closure.

So $5 \mid (y-x)$; thus yRx.

Hence R is symmetric, as claimed. ∎

Ex: Consider the relation depicted below:



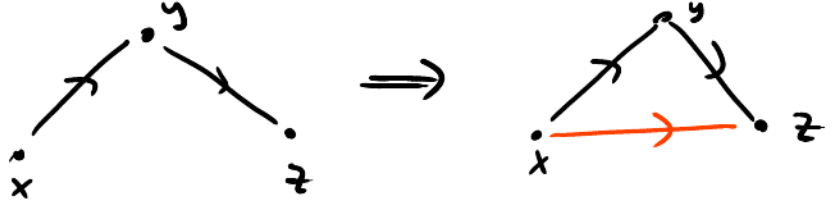
This relation is NOT symmetric: we have $1 R 2$, but $2 \not R 1$. ∎

On the other hand, R is reflexive.

Point: Reflexive does NOT imply symmetric...

Observation: if $x = y$ and $y = z$, then $x = z$.

Defn: A relation R on set S is transitive when for all $x, y, z \in S$ we have xRy and yRz implies xRz.

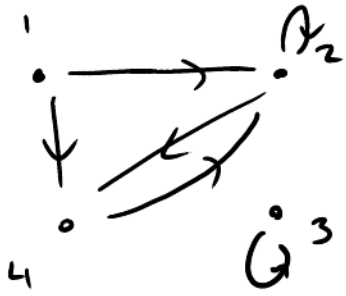
Picture: If you see 

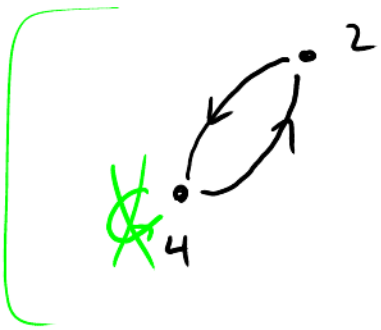
Ex: For xRy when $5|(x-y)$ on $\mathbb{Z}$, we note R is transitive. Let $x,y,z \in \mathbb{Z}$ and suppose xRy and yRz. Hence, $5|(x-y)$ and $5|(y-z)$ by definition of R. Thus there are $k,m \in \mathbb{Z}$ with $x-y=5k$ and $y-z=5m$. Now we notice

$$x-z = (x-y)+(y-z) = 5k+5m = 5(k+m).$$

As $k+m \in \mathbb{Z}$ by closure of $\mathbb{Z}$ under addition, we see $5|(x-z)$. Hence, by definition of R, we have xRz. Hence we have shown that for all $x,y,z \in \mathbb{Z}$, if both xRy and yRz, then xRz. Hence R is transitive by definition of transitivity. ☑

Ex: The relation depicted below is not transitive.





Indeed, ★
$4R2$ and
$2R4$, but
$4 \not R 4$.

Ex: is reflexive, symmetric and transitive.

$R = \{(3,3)\}$

Reflexive: ✓

Symmetric: $xRy \Rightarrow x=y=3 \Rightarrow yRx$.

Transitive: $xRy \wedge yRz \Rightarrow x=y=z=3 \Rightarrow xRz$. ☑

Q: Is $\cdot_x$ reflexive? Symmetric? Transitive?